N-Fractional calculus for ordinary differential equations of n-th order

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Abstract

There are many papers for application of fractional calculus and Shehu transform to fractional differential equations (see [5], [8], [10] and [11]). In this paper, we transform the ordinary differential equations of n-th order to fractional differential equations, then apply N-fractional calculus to get the solutions. Also, we get some special cases.

Key words: (N-fractional calculus, the non-homogeneous $n-th$ ordinary differential equation, the homogeneous $n-th$ order ordinary differential equation, fractional differential equation).

الملخص

في هذا البحث استخدمنا طريقة نيشوموتو لحل المعادلات التفاضلية العادية المتجانسة وغير المتجانسة من الرتبة النونية وذلك بتحويلهن إلى معادلات تفاضلية كسرية، ومن خلال هذه النتائج حصلنا على حالات خاصة تمثل نظريات لنيشوموتو والهاشمي لمعادلات تفاضلية عادية متجانسة وغير متجانسة من الرتبة الثانية.

الكلمات المفتاحية: (حساب التفاضل والتكامل الكسري لنيشيموتو، ومعادلة تفاضلية عادية غير متجانسة من الرتبة النونية، ومعادلة تفاضلية عادية متجانسة من الرتبة النونية، ومعادلة تفاضلية كسرية).

1. Introduction:

K. Nishimoto defined the fractional calculus of order (υ) for functions of single variable as the following (see [6]):

Definition:

Let $D=\{D_-,\ D_+\}$, $C=\{\ C_-\ ,\ C_+\}$, where

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-,

D_+ be a domain surrounded by C_+.

(Here D contains the points over curves C).

Moreover, let $f=f(z)$ be a regular function in D ($z\in D$),

$$f_\upsilon={}_C(f)_\nu(z)=\frac{\Gamma(\upsilon+1)}{2\pi i}\int_C\frac{f(\varsigma)}{(\varsigma-z)^{\upsilon+1}}d\varsigma,\qquad(\upsilon\notin Z^-)\qquad(1.1)$$

$$(f)_{-m}=\lim_{\nu\to-m}(f)_\nu\qquad(m\in Z^+),$$

where $\varsigma\neq z$, $z\in C$, $\upsilon\in R$, Γ:Gamma function, $-\pi\leq$ arg $(\varsigma-z)$ $\leq\pi$ for C_-,

$0\leq$ arg $(\varsigma-z)$ $\leq 2\pi$ for C_+,

then $(f)_\upsilon$ is the fractional differintegration of arbitrary order υ (derivatives of order υ for υ>0, and integrals of order $-\upsilon$ for $\upsilon<0$), with respect to z, of the function f, if $|(f)_\upsilon|<\infty$.

Shih-Tong Tu solved ordinary and partial differential equations of n-th order by using N-fractional calculus (see [9] and [10]). In [3] and [4] Bassim used

Nishimoto's method to solving ordinary and partial differential equations of second, third, fourth, and fifth order. Nishimoto applied N-fractional calculus to ordinary and partial differential equations of second, third, fourth and fifth order (see [6] and [7]). For example, Nishimoto introduced the following results (see [6], Vol. 1, p. 154-158):

Result (1): If f_α exists and $f_\alpha \neq 0$, then the differential equation

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot 2 \cdot \alpha \cdot z + \varphi \cdot \alpha \cdot (\alpha - 1) = f, \qquad (z \neq 0) \qquad (1.2)$$

has a particular solution of the form

$$\varphi = \left(f_{-\alpha} \cdot z^{-2}\right)_{\alpha-2}, \qquad (1.3)$$

where $\varphi = \varphi(z)$ and $\alpha, z \in C$.

Result (2): the differential equation

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot 2 \cdot \alpha \cdot z + \varphi \cdot \alpha \cdot (\alpha - 1) = 0, \qquad (z \neq 0) \qquad (1.4)$$

has a solution of the form

$$\varphi = \left(k \cdot z^{-2}\right)_{\alpha-2}, \qquad (1.5)$$

for $\alpha \notin Z^- \cup \{0\}$ **and** k is constant.

Result (3): If f_α exists and $f_\alpha \neq 0$, then the differential equation (1.2) has the solution

$$\varphi = \left(f_{-\alpha} \cdot z^{-2}\right)_{\alpha-2} + \left(k \cdot z^{-2}\right)_{\alpha-2}, \qquad (1.6)$$

for $\alpha \notin Z^- \cup \{0\}$.

Also, Al-Hashmi, A. M. H. solved ordinary differential equations of second and fourth order by using N-fractional calculus (see [1] and [2]). For example, he got the following results (see [2]):

Result (4): If f_α exists and $f_\alpha \neq 0$, then the differential equation

$$\varphi_2 \cdot (z^2 - z) + \varphi_1 \cdot (2 \cdot \alpha \cdot z - \alpha) + \varphi \cdot \alpha \cdot (\alpha - 1) = f, \qquad (z \neq 0,1) \qquad (1.7)$$

has a particular solution of the form

$$\varphi = \left(f_{-\alpha} \cdot \frac{1}{(z^2 - z)}\right)_{\alpha-2}, \qquad (1.8)$$

where $\varphi = \varphi(z)$ and $\alpha, z \in C$.

Result (5): the differential equation

$$\varphi_2 \cdot (z^2 - z) + \varphi_1 \cdot (2 \cdot \alpha \cdot z - \alpha) + \varphi \cdot \alpha \cdot (\alpha - 1) = 0, \qquad (z \neq 0,1) \tag{1.9}$$

has a solution of the form

$$\varphi = (k \cdot z + h)_{\alpha-2}, \tag{1.10}$$

where k and h are constants.

The main object of this paper is to apply N-fractional calculus to ordinary differential equations of n-th order and get some special cases.

By using (1.1) we get the following Lemmas:

Lemma 1.: If k is constant, then

$$(k)_\upsilon = k \cdot (1)_\upsilon = 0, \qquad \text{if} \quad \upsilon \notin Z^- \cup \{0\} \tag{1.11}$$

Lemma 2.: Let $u = u(z)$, $y = y(z)$ be regular functions and if u_v, y_v exist, then

$$(au + by)_\upsilon = a(u)_\upsilon + b(y)_\upsilon, \text{ where } a,b \text{ are constants } (z, \upsilon \in C). \tag{1.12}$$

Lemma 3.: Let $f = f(z)$ be regular function, if $(f_\mu)_\upsilon$ and $(f_\upsilon)_\mu$ exist, then

$$\left(f_\mu(z)\right)_\upsilon = \frac{\Gamma(\upsilon+1)}{2\pi i} \int_c \frac{f_\mu(\zeta)}{(\zeta - z)^{\upsilon+1}} d\zeta, \qquad (z, \upsilon, \mu \in C)$$

$$\left(f_\mu(z)\right)_\upsilon = \left(f_\upsilon(z)\right)_\mu$$

$$\left(f_\mu(z)\right)_\upsilon = (f)_{\upsilon+\mu}. \tag{1.13}$$

Lemma 4.: Let $u = u(z)$, $y = y(z)$ be regular functions and if are u_α, y_α exist, then

$$(u\, y)_\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - n + 1)\Gamma(n+1)} (y)_{\alpha-n} (u)_n, \qquad (z, \alpha \in C). \tag{1.14}$$

2.N-fractional calculus to n-th order ordinary differential equations

In this section, we introduce and prove the below results, but we can not solve differential equations of natural order by using N-fractional calculus, so that we must change the differential equations of natural order to fractional order as the following:

Theorem (1): If f_α exists and $f_\alpha \neq 0$, then the non-homogeneous $n-th$ order linear ordinary differential equation

$$\sum_{k=0}^{n} \varphi_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - k + 1) \cdot \Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right) = f, \qquad (a_1 \cdot z \neq 0) \tag{2.1}$$

has a particular solution of the form

$$\varphi = w_\alpha = \left(f_{-\alpha} \cdot \frac{1}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n}, \tag{2.2}$$

where $\varphi = \varphi(z)$, $f = f(z)$, $z \in C$ and $a_1, a_2, a_3, \cdots, a_{n+1}$ are constants.

Proof theorem (1):

Put $\varphi_n = w_{\alpha+n}$, where $\alpha \in]0,1[$ and $w = w(z)$. (2.3)

Substituting (2.3) into (2.1); we get

$$\sum_{k=0}^{n} w_{\alpha+n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right) = f. \tag{2.4}$$

The equation (2.4) is fractional differential equation of order $n+\alpha$.

Now, by using (1.14), then (2.4) becomes

$$\left(w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right)_{\alpha} = f,$$

By using (1.13); we get

$$w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r} = f_{-\alpha},$$

this is,

$$w_n = \frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}}.$$

By taking the order $(\alpha - n)$ of both sides; we obtain

$$w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}}\right)_{\alpha-n},$$

hence,

$$\varphi = w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}}\right)_{\alpha-n}.$$

as solution of the non-homogeneous $n-th$ order linear ordinary differential equation (2.1). Here completes the proof of the theorem (1).

Inversely, we have

$$\varphi = w_{\alpha} = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} . \tag{2.5}$$

Substituting (2.5) into the left hand side of (2.1); we get

$$L.H.S.(2.1) = \sum_{k=0}^{n} \left(\left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} \right)_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

By using (1.13); we obtain

$$\text{L. H. S. (2.1)} = \sum_{k=0}^{n} \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right). \tag{2.6}$$

Using (1.14), then (2.6) becomes

$$\textbf{L. H. S. (2.1)} = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_{\alpha} ,$$

L. H. S. (2.1) = f = R. H. S. (2.1).

Theorem (2): The homogeneous nth order linear ordinary differential equation

$$\sum_{k=0}^{n} \varphi_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right) = 0, \qquad (a \cdot z \neq 0) \tag{2.7}$$

has a solution of the form

$$\varphi = w_{\alpha} = \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} , \tag{2.8}$$

for $\alpha \notin Z^{-} \cup \{0\}$, $\varphi = \varphi(z)$, $z \in C$ and $m, a_1, a_2, a_3, \cdots, a_{n+1}$ are constants.

Proof theorem (2):

Here, we prove to the homogeneous nth order linear ordinary differential equation (2.7) its solution is (2.8) as the following:

Put $\varphi_n = w_{\alpha+n}$ in (2.7), we get:

$$\sum_{k=0}^{n} w_{\alpha+n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right) = 0, \tag{2.9}$$

Now, by using (1.14), then (2.9) becomes

$$\left(w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right)_{\alpha} = 0.$$

By applying (1.13); we get:

$$\left(\left(w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right)_1\right)_{\alpha-1} = 0,$$

and

$$\left(w_{n+1} \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r} + w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot (n-r)\cdot z^{n-r-1}\right)_{\alpha-1} = 0, \tag{2.10}$$

Now, by using (1.14) and (1.11), then (2.10) becomes

$$w_{n+1} \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r} + w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot (n-r)\cdot z^{n-r-1} = 0,$$

and

$$w_{n+1} \cdot \sum_{r=0}^{n} a_{r+1}\cdot z^{n-r} = -w_n \cdot \sum_{r=0}^{n} a_{r+1}\cdot (n-r)\cdot z^{n-r-1},$$

hence

$$\frac{w_{n+1}}{w_n} = -\frac{\sum_{r=0}^{n} a_{r+1}\cdot (n-r)\cdot z^{n-r-1}}{\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}}, \tag{2.11}$$

integrating both sides of (2.11), we get:

$$\log(w_n) = -\log\left(\sum_{r=0}^{n} a_{r+1}\cdot z^{n-r}\right) + \log(m),$$

this implies that

$$w_n = \frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}},$$

that is

$$w = \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{-n},$$

hence

$$\varphi = w_\alpha = \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n}.$$

which completes the proof of the theorem (2).

Inversely, we have

$$\varphi = w_\alpha = \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n}, \qquad (2.12)$$

Substituting (2.12) into the left hand side of (2.7); we get

$$\text{L. H. S.(2.7)}= \sum_{k=0}^{n} \left(\left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} \right)_{n-k} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

By using (1.13); we obtain

$$\text{L. H. S. (2.7)} = \sum_{k=0}^{n} \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

(2.13)

Using (1.14), then (2.13) becomes

L. H. S. (2.7) = $\left(\dfrac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_{\alpha}$,

Now, by using (1.11), then

L. H. S. (2.7) $= 0 =$ R. H. S. (2.7)

Theorem (3): If f_{α} exists and $f_{\alpha} \neq 0$, then the differential equation (2.1) has the solution

$$\varphi = w_{\alpha} = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} , \qquad (2.14)$$

for $\alpha \notin Z^{-} \cup \{0\}$.

Proof theorem (3):

Put $\varphi_n = w_{\alpha+n}$, (2.15)

Substituting (2.15) into (2.1); we get

$$\sum_{k=0}^{n} w_{\alpha+n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1) \cdot \Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right) = f . \qquad (2.16)$$

Now, by using (1.14) and (1.11) for $\alpha \notin Z^{-} \cup \{0\}$, then (2.16) becomes

$$\left(w_n \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_{\alpha} = f + (m)_{\alpha} ,$$

By using (1.12) and (1.13); we get

$$w_n \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} = f_{-\alpha} + m,$$

this is,

$$w_n = \frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} + \frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}}$$

By taking the order $(\alpha - n)$ of both sides; we obtain

$$w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} + \frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n}$$

hence,

$$\varphi = w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right). \qquad (2.17)$$

which completes the proof of the theorem (3).

Inversely, we have

Substituting (2.17) in the left hand side of (2.1); we get:

L. H. S. (2.1) =

$$\sum_{k=0}^{n} \left(\left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\frac{m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} \right)_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

By using (1.12); we obtain

$$\text{L.H.S.(2.1)}= \sum_{k=0}^{n} \left(\left(\frac{f_{-\alpha}+m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} \right)_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

By using (1.13); we obtain

$$\text{L. H. S. (2.1)} = \sum_{k=0}^{n} \left(\frac{f_{-\alpha}+m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k}\left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right), \qquad (2.18)$$

Using (1.14), then (2.18) becomes

$$\text{L. H. S. (2.1)} = \left(\frac{f_{-\alpha}+m}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_{\alpha},$$

L. H. S. (2.1) $=\left(f_{-\alpha}+m\right)_{\alpha}$,

L. H. S. (2.1) = $f =$ R. H. S. (2.1)

Theorem (4): The homogeneous nth order linear ordinary differential equation

$$\sum_{k=0}^{n}\varphi_{n-k}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\cdot\frac{d^k}{dz^k}\left(\sum_{r=0}^{n}a_{r+1}\cdot z^{n-r}\right)=0, \qquad (\ a\cdot z\neq 0\ ) \qquad (2.19)$$

has other solution of the form

$$\varphi=w_{\alpha}=\left(\sum_{r=1}^{n}b_r\cdot\frac{z^{n-r}}{(n-r)!}\right)_{\alpha}, \qquad (2.20)$$

where $\varphi=\varphi(z)$, $z\in C$ and $b_1,b_2,b_3,\ \cdots,b_n$ are constants.

Proof theorem (4):

Now we will prove theorem (4) as the following:

Put $\varphi_n=w_{\alpha+n}$ in (2.19) we get:

$$\sum_{k=0}^{n}w_{\alpha+n-k}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\cdot\frac{d^k}{dz^k}\left(\sum_{r=0}^{n}a_{r+1}\cdot z^{n-r}\right)=0, \qquad (2.21)$$

Now, by using (1.14), then (2.21) becomes

$$\left(w_n\cdot\sum_{r=0}^{n}a_{r+1}\cdot z^{n-r}\right)_{\alpha}=0,$$

and,

$$w_n\cdot\sum_{r=0}^{n}a_{r+1}\cdot z^{n-r}=0,$$

this implies that,

$$w_n=0,$$

then,

$$w_{n-1}=b_1,$$

$$w_{n-2}=b_1z+b_2,$$

$$w_{n-3}=b_1\frac{z^2}{2!}+b_2z+b_3,$$

$$w_{n-4} = b_1 \frac{z^3}{3!} + b_2 \frac{z^2}{2!} + b_3 z + b_4,$$

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$$w_{n-n} = b_1 \frac{z^{n-1}}{(n-1)!} + b_2 \frac{z^{n-2}}{(n-2)!} + b_3 \frac{z^{n-3}}{(n-3)!} + \cdots\cdots + b_n,$$

hence,

$$w = \sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!}. \qquad (2.22)$$

By taking the order (α) to both sides of (2.22); we get

$$\varphi = w_\alpha = \left(\sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!} \right)_\alpha .$$

which completes the proof of theorem (4).

Inversely, we have

$$\varphi = w_\alpha = \left(\sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!} \right)_\alpha , \qquad (2.23)$$

Substituting (2.23) into the left hand side of (2.19); we get

$$\text{L. H. S.}(2.19) = \sum_{k=0}^{n} \left(\left(\sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!} \right)_\alpha \right)_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

By using (1.13); we obtain

$$\text{L. H. S.}(2.19) = \sum_{k=0}^{n} \left(\left(\sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!} \right)_n \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

$$(2.24)$$

Using (1.14), then (2.24) becomes

$$\text{L. H. S. } (2.19) = \left(\left(\sum_{r=1}^{n} b_r \frac{z^{n-r}}{(n-r)!} \right)_n \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_\alpha .$$

L. H. S. (2.19) =0,

since $(z^n)_k = 0,\ if\ k > n\ \ and\ \ n,k \in Z^+$.

hence,

L. H. S. (2.19) $= 0 =$ R. H. S. (2.19)

Theorem (5): If f_α exists and $f_\alpha \neq 0$, then the differential equation (2.1) has the solution

$$\varphi = w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_\alpha , \tag{2.25}$$

Proof theorem (5):

Put $\varphi_n = w_{\alpha+n}$, (2.26)

Substituting (2.26) into (2.1); we get

$$\sum_{k=0}^{n} w_{\alpha+n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right) = f. \tag{2.27}$$

Now, by using (1.14), then (2.27) becomes

$$\left(w_n \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_\alpha = f,$$

By using (1.13); we get

$$w_n \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} = f_{-\alpha},$$

this is,

$$w_n = \frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_n$$

By taking the order $(\alpha - n)$ of both sides; we obtain

$$w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_n \right)_{\alpha-n}$$

hence,

$$\varphi = w_\alpha = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_\alpha . \qquad (2.28)$$

which completes the proof.

Inversely, we have

Substituting (2.28) in the left hand side of (2.1); we get:

L. H. S. (2.1) =

$$\sum_{k=0}^{n} \left(\left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-n} + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_\alpha \right)_{n-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

(2.29)

By using (1.12) and (1.13), then (2.29) becomes

L. H. S. (2.1) =

$$\sum_{k=0}^{n} \left(\left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right) + \left(\sum_{r=1}^{n} b_r \cdot \frac{z^{n-r}}{(n-r)!} \right)_n \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$ L. H. S. (2.1)=

$$\sum_{k=0}^{n} \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \right)_{\alpha-k} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \cdot \frac{d^k}{dz^k} \left(\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right),$$

(2.30)

Using (1.14), then (2.30) becomes

$$\text{L. H. S. (2.1)} = \left(\frac{f_{-\alpha}}{\sum_{r=0}^{n} a_{r+1} \cdot z^{n-r}} \cdot \sum_{r=0}^{n} a_{r+1} \cdot z^{n-r} \right)_\alpha ,$$

L. H. S. (2.1) = f = R. H. S. (2.1)

3. Conclusions:

Finally, we show some special cases from this work as the following:

(i) Put $n = 2,\ a_1 = 1$ and $a_i = 0$ for $\forall i \in \{2,3,4,5,......,n+1\}$ in theorems (1), (2) and (3) respectively, we get the results (1), (2) and (3) respectively.

(ii) Put $n = 2$, $a_1 = 1$, $a_2 = -1$ and $a_i = 0$ for $\forall i \in \{3,4,5,......,n+1\}$ in theorems

(1) and (4) respectively, we get the results (4) and (5) respectively.

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